UNIT 2: REASONING WITH LINEAR EQUATIONS AND INEQUALITIES

This unit investigates linear equations and inequalities. Students create linear equations and inequalities and use them to solve problems. They learn the process of reasoning and justify the steps used to solve simple equations. Students also solve systems of equations and represent linear equations and inequalities graphically. They write linear functions that describe a relationship between two quantities and write arithmetic sequences recursively and explicitly. They understand the concept of a function and use function notation. Given tables, graphs, and verbal descriptions, students interpret key characteristics of linear functions and analyze these functions using different representations.

Solving Equations and Inequalities in One Variable

MGSE9-12.A.REI.3 Solve linear equations and inequalities in one variable including equations with coefficients represented by letters. For example, given ax + 3 = 7, solve for x.

Solve systems of equations.

MGSE9-12.A.REI.5 Show and explain why the elimination method works to solve a system of two-variable equations.

MGSE9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

KEY IDEAS

1. Solving an equation or inequality means finding the quantity or quantities that make the equation or inequality true. The strategies for solving an equation or inequality depend on the number of variables and the exponents that are included.

2. Here is an algebraic method for solving a linear equation with one variable:

   Apply algebraic properties to write equivalent expressions until the desired variable is isolated on one side. Be sure to check your answers.

Example:

Solve 2(3 – a) = 18.

Solution:

Solve the equation using either of these two ways:

\[
\begin{align*}
2(3 – a) &= 18 \\
3 – a &= 9 \\
-a &= 6 \\
a &= -6
\end{align*}
\]

\[
\begin{align*}
2(3 – a) &= 18 \\
6 - 2a &= 18 \\
-2a &= 12 \\
a &= -6
\end{align*}
\]

Distributive Property

Addition Property of Equality

Multiplicative Inverses and Identity Property of 1
3. Here is an algebraic method for solving a linear inequality with one variable:
Write equivalent expressions until the desired variable is isolated on one side. If you multiply or divide by a negative number, make sure you reverse the inequality symbol.

**Example:**

Solve $2(5 - x) > 8$ for $x$.

**Solution:**

Solve the inequality using either of these two ways:

\[
\begin{align*}
2(5 - x) & > 8 \\
5 - x & > 4 \\
10 - 2x & > 8 \\
-2x & > -2 \\
x & < 1 \\
x & < 1
\end{align*}
\]

**Distributive Property**

**Addition Property of Inequality**

**Multiplicative Inverses and Identity Property of 1**

**Important Tips**

- If you multiply or divide both sides of an inequality by a negative number, make sure you reverse the inequality sign.
- Be familiar with the properties of equality and inequality so you can transform equations or inequalities.
  - The *addition property* of equality tells us that adding the same number to each side of an equation gives us an equivalent equation.
    
    Example: if $a - b = c$, then $a - b + b = c + b$, or $a = c + b$
  
  - The *multiplication property* of equality tells us that multiplying the same number to each side of an equation gives us an equivalent equation.
    
    Example: if $\frac{a}{b} = c$, then $\frac{a}{b} \cdot b = c \cdot b$, or $a = c \cdot b$

  - The *multiplication inverse property* tells us that multiplying a number by its reciprocal equals 1.
    
    Example: $\frac{1}{a}(a) = 1$

  - The *additive inverse property* tells us that adding a number to its opposite equals 0.
    
    Example: $a + (-a) = 0$

- Sometimes eliminating denominators by multiplying all terms by a common denominator or common multiple makes it easier to solve an equation or inequality.
REVIEW EXAMPLES

1. Karla wants to save up for a prom dress. She figures she can save $9 each week from the money she earns babysitting. If she plans to spend less than $150 for the dress, how many weeks will it take her to save enough money to buy any dress in her price range?

Solution:
Let \( w \) represent the number of weeks. If she saves $9 each week, Karla will save 9\( w \) dollars after \( w \) weeks. We need to determine the minimum number of weeks it will take her to save $150. Use the inequality 9\( w \) \( \geq \) 150 to solve the problem. We need to transform 9\( w \) \( \geq \) 150 to isolate \( w \). Divide both sides by 9 to get \( w \geq 16 \frac{2}{3} \) weeks.
Because we do not know what day Karla gets paid each week, we need the answer to be a whole number. So, the answer has to be 17, the smallest whole number greater than 16 \( \frac{2}{3} \). She will save $144 after 16 weeks and $153 after 17 weeks.

2. Joachim wants to know if he can afford to add texting to his cell phone plan. He currently spends $21.49 per month for his cell phone plan, and the most he can spend for his cell phone is $30 per month. He could get unlimited text messaging added to his plan for an additional $10 each month. Or, he could get a “pay-as-you-go” plan that charges a flat rate of $0.15 per text message. He assumes that he will send an average of 5 text messages per day. Can Joachim afford to add a text message plan to his cell phone plan?

Solution:
Joachim cannot afford either plan.

At an additional $10 per month for unlimited text messaging, Joachim’s cell phone bill would be $31.49 a month. If he chose the pay-as-you-go plan, each day he would expect to pay for 5 text messages. Let \( t \) stand for the number of text messages per month. Then, on the pay-as-you-go plan, Joachim could expect his cost to be represented by the expression:

\[
21.49 + 0.15t
\]
If he must keep his costs at $30 or less, \( 21.49 + 0.15t \leq 30 \).

To find the number of text messages he can afford, solve for \( t \).

\[
21.49 – 21.49 + 0.15t \leq 30 – 21.49 \quad \text{Subtract 21.49 from both sides.}
\]

\[
0.15t \leq 8.51 \quad \text{Combine like terms.}
\]

\[
t \leq 56.733 \ldots \quad \text{Divide both sides by 0.15.}
\]

The transformed inequality tells us that Joachim would need to send fewer than 57 text messages per month to afford the pay-as-you-go plan. However, 5 text messages per day at a minimum of 28 days in a month is 140 text messages per month. So, Joachim cannot afford text messages for a full month, and neither plan fits his budget.
3. Two cars start at the same point and travel in opposite directions. The first car travels 15 miles per hour faster than the second car. In 4 hours, the cars are 300 miles apart. Use the formula below to determine the rate of the second car.

\[ 4(r + 15) + 4r = 300 \]

What is the rate, \( r \), of the second car?

**Solution:**
The second car is traveling 30 miles per hour.

\[ 4(r + 15) + 4r = 300 \]  
Write the original equation.

\[ 4r + 60 + 4r = 300 \]  
Multiply 4 by \( r + 15 \).

\[ 8r + 60 = 300 \]  
Combine like terms.

\[ 8r = 240 \]  
Subtract 60 from each side.

\[ r = 30 \]  
Divide each side by 8.

4. Solve the equation \( 14 = ax + 6 \) for \( x \). Show and justify your steps.

**Solution:**

\[ 14 = ax + 6 \]  
Write the original equation.

\[ 14 - 6 = ax + 6 - 6 \]  
Subtraction Property of Equality

\[ 8 = ax \]  
Combine like terms on each side.

\[ \frac{8}{a} = \frac{ax}{a} \]  
Division Property of Equality; \( a \neq 0 \)

\[ \frac{8}{a} = x \]  
Simplify.
SAMPLE ITEMS

1. This equation can be used to find $h$, the number of hours it will take Flo and Bryan to mow their lawn.

   \[
   \frac{h}{3} + \frac{h}{6} = 1
   \]

   How many hours will it take them to mow their lawn?
   
   A. 6 hours  
   B. 3 hours  
   C. 2 hours  
   D. 1 hour

   Correct Answer: C

2. A ferry boat carries passengers back and forth between two communities on the Peachville River.

   - It takes 30 minutes longer for the ferry to make the trip upstream than downstream.
   - The ferry’s average speed in still water is 15 miles per hour.
   - The river’s current is usually 5 miles per hour.

   This equation can be used to determine how many miles apart the two communities are.

   \[
   \frac{m}{15 - 5} = \frac{m}{15 + 5} + 0.5
   \]

   What is $m$, the distance between the two communities?
   
   A. 0.5 mile  
   B. 5 miles  
   C. 10 miles  
   D. 15 miles

   Correct Answer: C
3. For what values of $x$ is the inequality $\frac{2}{3} + \frac{x}{3} > 1$ true?

A. $x < 1$
B. $x > 1$
C. $x < 5$
D. $x > 5$

Correct Answer: B

4. Look at the steps used when solving $3(x - 2) = 3$ for $x$.

\[
\begin{align*}
3(x - 2) &= 3 & \text{Write the original equation.} \\
3x - 6 &= 3 & \text{Use the Distributive Property.} \\
3x - 6 + 6 &= 3 + 6 & \text{Step 1} \\
x &= 9 & \text{Step 2} \\
\frac{3x}{3} &= \frac{9}{3} & \text{Step 3} \\
x &= 3 & \text{Step 4}
\end{align*}
\]

Which step is the result of combining like terms?

A. Step 1
B. Step 2
C. Step 3
D. Step 4

Correct Answer: B
Solving a System of Two Linear Equations

MGSE9-12.A.REI.5 Show and explain why the elimination method works to solve a system of two-variable equations.

MGSE9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

KEY IDEAS

1. A system of linear equations consists of two or more linear equations that may or may not have a common solution. The solution of a system of two linear equations is the set of values for the variables that makes all the equations true. The solutions can be expressed as ordered pairs \((x, y)\) or as two equations, one for \(x\) and the other for \(y\) \((x = \ldots\) and \(y = \ldots\)).

Strategies:

* Use tables or graphs as strategies for solving a system of equations. For tables, use the same values for both equations. For graphs, the intersection of the graph of both equations provides the solution to the system of equations.

Example:
Solve this system of equations.

\[
\begin{align*}
  y &= 2x - 4 \\
  x &= y + 1
\end{align*}
\]

Solution:
First, find coordinates of points for each equation. Making a table of values for each is one way to do this. Use the same values for both equations.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(2x - 4)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>−1</td>
<td>−6</td>
<td>−1</td>
<td>−2</td>
</tr>
<tr>
<td>0</td>
<td>−4</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>1</td>
<td>−2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
Graph the first equation by using the $y$-intercept, $(0, -4)$, and the slope, 2. Graph the second equation by solving for $y$ to get $y = x - 1$ and then use the $y$-intercept, $(0, -1)$, and the slope, 1. Both equations are displayed on the graph below.

The graph shows all the ordered pairs of numbers (rows from the table), that satisfy $y = 2x - 4$ and the ordered pairs that satisfy $x = y + 1$. From the graph, it appears that the lines cross at about $(3, 2)$. Then try that combination in both equations to determine whether $(3, 2)$ is a solution to both equations.

\[
\begin{align*}
  y &= 2x - 4 \\
  x &= y + 1 \\
  (2) &= 2(3) - 4 \\
  2 &= 6 - 4 \\
  2 &= 2
\end{align*}
\]

So, $(3, 2)$ is the solution to the system of equations. The graph also suggests that $(3, 2)$ is the only point the lines have in common, so we have found the only pair of numbers that works for both equations.

**Strategies:**

* Simplify the problem by eliminating one of the two variables.

**Substitution method:** Use one equation to isolate a variable and replace that variable in the other equation with the equivalent expression you just found. Solve for the one remaining variable. Use the solution to the remaining variable to find the unknown you eliminated.
Example:
Solve this system of equations.

\[
\begin{align*}
2x - y &= 1 \\
5 - 3x &= 2y
\end{align*}
\]

Solution:
Begin by choosing one of the equations and solving for one of the variables. This variable is the one you will eliminate. We could solve the top equation for \( y \).

\[
2x - y = 1
\]
\[
2x = 1 + y
\]
\[
2x - 1 = y
\]
\[
y = 2x - 1
\]

Next, use substitution to replace the variable you are eliminating in the other equation.

\[
5 - 3x = 2y
\]
\[
5 - 3x = 2(2x - 1)
\]
\[
5 - 3x = 4x - 2
\]
\[
7 = 7x
\]
\[
x = 1
\]

Now, find the corresponding \( y \)-value. You can use either equation.

\[
2x - y = 1
\]
\[
2x - 1 = y
\]
\[
2 - y = 1
\]
\[
-y = 1 - 2
\]
\[
-y = -1
\]
\[
y = 1
\]

So, the solution is \( x = 1 \) and \( y = 1 \), or \((1, 1)\).

Check solution:

\[
\begin{align*}
2x - y &= 1 \\
2(1) - (1) &= 1 \\
2 - 1 &= 1
\end{align*}
\]
\[
\begin{align*}
5 - 3x &= 2y \\
5 - 3(1) &= 2(1) \\
5 - 3 &= 2
\end{align*}
\]
\[
1 = 1 \\
2 = 2
\]

Elimination method: Add the equations (or a transformation of the equations) to eliminate a variable. Then solve for the remaining variable and use this value to find the value of the variable you eliminated.
Example:
Solve this system of equations.
\[
\begin{align*}
2x - y &= 1 \\
5 - 3x &= -y
\end{align*}
\]

Solution:
First, rewrite the second equation in standard form.
\[
\begin{align*}
2x - y &= 1 \\
-3x + y &= -5
\end{align*}
\]

Decide which variable to eliminate. We can eliminate the \( y \)-terms because they are opposites.
\[
\begin{align*}
2x - y &= 1 \\
-3x + y &= -5
\end{align*}
\]

Add the equations, term by term, eliminating \( y \) and reducing to one equation. This is an application of the addition property of equality.
\[
-x = -4
\]

Multiply both sides by \(-1\) to solve for \( x \). This is an application of the multiplication property of equality.
\[
(-1)(-x) = (-1)(-4) \\
x = 4
\]

Now substitute this value of \( x \) in either original equation to find \( y \).
\[
\begin{align*}
2x - y &= 1 \\
2(4) - y &= 1 \\
8 - y &= 1 \\
y &= -7
\end{align*}
\]

The solution to the system of equations is \((4, 7)\).

Check solution:
\[
\begin{align*}
2x - y &= 1 & -3x + y &= -5 \\
2(4) - (7) &= 1 & -3(4) + (7) &= -5 \\
8 - 7 &= 1 & -12 + 7 &= -5 \\
1 &= 1 & -5 &= -5
\end{align*}
\]
Example:
Solve this system of equations.
\[
\begin{align*}
3x - 2y &= 7 \\
2x - 3y &= 3
\end{align*}
\]

Solution:
First, decide which variable to eliminate. We can eliminate the \(y\)-terms but will need to change the coefficients of the \(y\)-terms so that they are the same value or opposites. We can multiply the first equation by 3 and the second equation by \(-2\). This is the multiplication property of equality.

\[
\begin{align*}
(3)(3x - 2y &= 7) \rightarrow 9x - 6y = 21 \\
(2)(2x - 3y &= 3) \rightarrow 4x - 6y = 6
\end{align*}
\]

Subtract the second equation from the first equation, term by term, eliminating \(y\) and reducing to one equation. This is the addition property of equality.

\[
x = 3
\]

Now substitute this value of \(x\) in either original equation to find \(y\).

\[
\begin{align*}
3x - 2y &= 7 \\
3(3) - 2y &= 7 \\
9 - 2y &= 7 \\
-2y &= -2 \\
y &= 1
\end{align*}
\]

The solution to the system of equations is \((3, 1)\).

Check solution:
\[
\begin{align*}
3x - 2y &= 7 & 2x - 3y &= 3 \\
3(3) - 2(1) &= 7 & 2(3) - 3(1) &= 3 \\
9 - 2 &= 7 & 6 - 3 &= 3 \\
7 &= 7 & 3 &= 3
\end{align*}
\]

2. The graphing method only suggests the solution of a system of equations. To check the solution, substitute the values into the equations and make sure the ordered pair satisfies both equations.

3. When graphing a system of equations:
   a. If the lines are parallel, then there is no solution to the system.
   b. If the lines coincide, then the lines have all their points in common and any pair of points that satisfies one equation will satisfy the other.
4. When using elimination to solve a system of equations, if both variables are removed when you try to eliminate one, and if the result is a true equation such as $0 = 0$, then the lines coincide. The equations would have all ordered pairs in common, as shown in the following graph.

![Graph showing two lines coinciding]

**Example:**
Solve this system of equations.

\[
\begin{align*}
3x - 3y &= 3 \\
x - y &= 1
\end{align*}
\]

**Solution:**
First, decide which variable to eliminate. We can eliminate the $y$-terms but will need to change the coefficients of the $y$-terms so that they are the same value or opposites. We can multiply the second equation by 3. This is the multiplication property of equality.

\[
\begin{align*}
(3x - 3y &= 3) \rightarrow 3x - 3y = 3 \\
(3)(x - y &= 1) \rightarrow 3x - 3y = 3
\end{align*}
\]

Subtract the second equation from the first equation, term by term, eliminating $y$ and reducing to one equation. This is the addition property of equality.

\[0 = 0\]

The solution to the system of equations is any value of $x$ that gives the same value of $y$ for either equation.

**Check solution:**
Substitute $x = 1$ in either original equation to find $y$.

\[
\begin{align*}
3x - 3y &= 3 \\
3(1) - 3y &= 3 \\
3 - 3y &= 3 \\
-3y &= 0 \\
y &= 0
\end{align*}
\]

A solution to the system of equations is $(1, 0)$. 
Unit 2: Reasoning with Linear Equations and Inequalities

Check solution:

\[
\begin{align*}
3x - 3y &= 3 \\
3(1) - 3(0) &= 3 \\
3 - 0 &= 3 \\
3 &= 3
\end{align*}
\]

\[
\begin{align*}
x - y &= 1 \\
(1) - (0) &= 1 \\
1 - 0 &= 1 \\
1 &= 1
\end{align*}
\]

When using elimination to solve a system of equations, if the result is a false equation such as 3 = 7, then the lines are parallel. The system of equations has no solution since there is no point where the lines intersect.

Example:

Solve this system of equations.

\[
\begin{align*}
3x - 3y &= 7 \\
x - y &= 1
\end{align*}
\]

Solution:

First, decide which variable to eliminate. We can eliminate the \(y\)-terms but will need to change the coefficients of the \(y\)-terms so that they are the same value or opposites. We can multiply the second equation by 3. This is the multiplication property of equality.

\[
\begin{align*}
(3x - 3y = 7) &\rightarrow 3x - 3y = 7 \\
(3)(x - y = 1) &\rightarrow 3x - 3y = 3
\end{align*}
\]

Subtract the second equation from the first equation, term by term, eliminating \(y\) and reducing to one equation. This is the addition property of equality.

\[
0 = 4
\]

The system of equations has no solutions.
REVIEW EXAMPLES

1. Consider the equations \( y = 2x - 3 \) and \( y = -x + 6 \).

   a. Complete the tables below.

   \[
   \begin{array}{cc}
   y = 2x - 3 & y = -x + 6 \\
   \hline
   x & y & x & y \\
   \hline
   -1 & -5 & -1 & 7 \\
   0 & -3 & 0 & 6 \\
   1 & -1 & 1 & 5 \\
   2 & 1 & 2 & 4 \\
   3 & 3 & 3 & 3 \\
   \end{array}
   \]

   b. Is there an ordered pair that satisfies both equations? If so, what is it?

   c. Graph both equations on the same coordinate plane by plotting the ordered pairs from the tables and connecting the points.

   d. Do the lines appear to intersect? If so, where? How can you tell that the point where the lines appear to intersect is a common point for both lines?

Solution:

   a. 

   \[
   \begin{array}{cc}
   y = 2x - 3 & y = -x + 6 \\
   \hline
   x & y & x & y \\
   \hline
   -1 & -5 & -1 & 7 \\
   0 & -3 & 0 & 6 \\
   1 & -1 & 1 & 5 \\
   2 & 1 & 2 & 4 \\
   3 & 3 & 3 & 3 \\
   \end{array}
   \]
b. Yes, the ordered pair (3, 3) satisfies both equations.

d. The lines appear to intersect at (3, 3). When $x = 3$ and $y = 3$ are substituted into each equation, the values satisfy both equations. This proves that (3, 3) lies on both lines, which means it is a common solution to both equations.

2. Rebecca has five coins worth 65 cents in her pocket. If she only has quarters and nickels, how many quarters does she have? Use a system of equations to arrive at your answer and show all steps.

**Solution:**
If $q$ represents the number of quarters and $n$ represents the number of nickels, the two equations could be $25q + 5n = 65$ (value of quarters plus value of nickels is 65 cents) and $q + n = 5$ (she has 5 coins). The equations in the system would be $25q + 5n = 65$ and $q + n = 5$.

Next, solve $q + n = 5$ for $q$. By subtracting $n$ from both sides, the result is $q = 5 - n$.

Next, eliminate $q$ by replacing $q$ with $5 - n$ in the other equation: $25(5 - n) + 5n = 65$.

Solve this equation for $n$.

\[
25(5 - n) + 5n = 65
\]
\[
125 - 25n + 5n = 65
\]
\[
125 - 20n = 65
\]
\[
-20n = -60
\]
\[
n = 3
\]

Now solve for $q$ by replacing $n$ with 3 in the equation $q = 5 - n$. So, $q = 5 - 3 = 2$, so 2 is the number of quarters.

Rebecca has 2 quarters and 3 nickels.
3. Peg and Larry purchased “no contract” cell phones. Peg’s phone costs $25 plus $0.25 per minute. Larry’s phone costs $35 plus $0.20 per minute. After how many minutes of use will Peg’s phone cost more than Larry’s phone?

**Solution:**

Let \( x \) represent the number of minutes used. Peg’s phone costs \( 25 + 0.25x \). Larry’s phone costs \( 35 + 0.20x \). We want Peg’s cost to exceed Larry’s.

This gives us \( 25 + 0.25x > 35 + 0.20x \), which we then solve for \( x \).

\[
25 + 0.25x > 35 + 0.20x \\
25 + 0.25x - 0.20x > 35 + 0.20x - 0.20x \\
25 + 0.05x > 35 \\
25 - 25 + 0.05x > 35 - 25 \\
0.05x > 10 \\
\frac{0.05x}{0.05} > \frac{10}{0.05} \\
x > 200
\]

After 200 minutes of use, Peg’s phone will cost more than Larry’s phone.

**Check solution:**

Since Peg’s phone will cost more than Larry’s phone after 200 minutes, we can substitute 201 minutes to check if it is true.

\[
25 + 0.25x > 35 + 0.20x \\
25 + 0.25(201) > 35 + 0.20(201) \\
25 + 50.25 > 35 + 40.2 \\
75.25 > 75.20
\]
4. Is \((3, -1)\) a solution of this system?
\[
\begin{align*}
y &= 2 - x \\
3 - 2y &= 2x
\end{align*}
\]

**Solution:**
Substitute the coordinates \((3, -1)\) into each equation.
\[
\begin{align*}
y &= 2 - x \\
3 - 2y &= 2x
\end{align*}
\]
\[
\begin{align*}
y &= 2 - 3 \\
3 - 2(-1) &= 2(3)
\end{align*}
\]
\[
\begin{align*}
-1 &= 2 - 3 \\
3 + 2 &= 6
\end{align*}
\]
\[
\begin{align*}
-1 &= -1 \\
5 &= 6
\end{align*}
\]

The coordinates of the given point do not satisfy \(3 - 2y = 2x\). If you get a false equation when trying to solve a system algebraically, then it means that the coordinates of the point are not the solution. So, \((3, -1)\) is not a solution of the system.

5. Solve this system.
\[
\begin{align*}
x - 3y &= 6 \\
-x + 3y &= -6
\end{align*}
\]

**Solution:**
Add the terms of the equations. Each pair of terms consists of opposites, and the result is \(0 + 0 = 0\).
\[
\begin{align*}
x - 3y &= 6 \\
-x + 3y &= -6
\end{align*}
\]
\[
\begin{align*}
0 &= 0
\end{align*}
\]

This result is always true, so the two equations represent the same line. Every point on the line is a solution to the system.
6. Solve this system.

\begin{align*}
-3x - y &= 10 \\
3x + y &= -8
\end{align*}

**Solution:**
Add the terms in the equations: \(0 = 2\).

The result is never true. The two equations represent parallel lines. As a result, the system has no solution.
1. Two lines are graphed on this coordinate plane.

Which point appears to be a solution of the equations of both lines?

A. (0, –2)  
B. (0, 4)  
C. (2, 0)  
D. (3, 1)

Correct Answer: D
2. Based on the tables, at what point do the lines $y = -x + 5$ and $y = 2x - 1$ intersect?

<table>
<thead>
<tr>
<th>$y = -x + 5$</th>
<th>$y = 2x - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

A. (1, 1)  
B. (3, 5)  
C. (2, 3)  
D. (3, 2)  

Correct Answer: C

3. Look at the tables of values for two linear functions, $f(x)$ and $g(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>16</td>
<td>-1</td>
<td>-18</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>0</td>
<td>-14</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>-10</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>5</td>
<td>-8</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>-14</td>
<td>7</td>
<td>14</td>
</tr>
</tbody>
</table>

What is the solution to $f(x) = g(x)$?

Solution:  
The solution to $f(x) = g(x)$ is $x = 3$. This is the value of $x$ where $f(x)$ and $g(x)$ both equal $-2$.  


4. Which ordered pair is a solution of \(3y + 2 = 2x - 5\)?

A. \((-5, 2)\)  
B. \((0, -5)\)  
C. \((5, 1)\)  
D. \((7, 5)\)

**Explanation of correct answer:** The correct answer is choice (C) \((5, 1)\). Also, if the values of \(x\) and \(y\) are substituted into the equation, the statement becomes \(5 = 5\), which is a true statement. This shows that the ordered pair is a solution of the equation.

**Correct Answer:** C

5. A manager is comparing the cost of buying baseball caps from two different companies.

- Company X charges a $50 fee plus $7 per baseball cap.
- Company Y charges a $30 fee plus $9 per baseball cap.

For what number of baseball caps will the cost be the same at both companies?

A. 10  
B. 20  
C. 40  
D. 100

**Correct Answer:** A

6. A shop sells one-pound bags of peanuts for $2 and three-pound bags of peanuts for $5. If 9 bags are purchased for a total cost of $36, how many three-pound bags were purchased?

A. 3  
B. 6  
C. 9  
D. 18

**Correct Answer:** B
7. Which graph represents a system of linear equations that has multiple common coordinate pairs?

Correct Answer: B
Represent and Solve Equations and Inequalities Graphically

**MGSE9-12.A.REI.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane.

**MGSE9-12.A.REI.11** Using graphs, tables, or successive approximations, show that the solution to the equation \( f(x) = g(x) \) is the \( x \)-value where the \( y \)-values of \( f(x) \) and \( g(x) \) are the same.

**MGSE9-12.A.REI.12** Graph the solution set to a linear inequality in two variables. Build a function that models a relationship between two quantities.

**KEY IDEAS**
1. The graph of a linear equation in two variables is a collection of ordered pair solutions in a coordinate plane. It is a graph of a straight line. Often tables of values are used to organize the ordered pairs.

**Example:**
Every year Silas buys fudge at the state fair. He buys two types: peanut butter and chocolate. This year he intends to buy $24 worth of fudge. If chocolate costs $4 per pound and peanut butter costs $3 per pound, what are the different combinations of fudge that he can purchase if he only buys whole pounds of fudge?

**Solution:**
If we let \( x \) be the number of pounds of chocolate and \( y \) be the number of pounds of peanut butter, we can use the equation \( 4x + 3y = 24 \). Now we can solve this equation for \( y \) to make it easier to complete our table.

\[
\begin{align*}
4x + 3y &= 24 \\
4x - 4x + 3y &= 24 - 4x \\
3y &= 24 - 4x \\
\frac{3y}{3} &= \frac{24 - 4x}{3} \\
y &= \frac{24 - 4x}{3}
\end{align*}
\]

Write the original equation.
Addition Property of Equality
Additive Inverse Property
Multiplication Property of Equality
Multiplicative Inverse Property
We will only use whole numbers in the table because Silas will only buy whole pounds of fudge.

<table>
<thead>
<tr>
<th>Chocolate (x)</th>
<th>Peanut butter (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>$6\frac{2}{3}$ (not a whole number)</td>
</tr>
<tr>
<td>2</td>
<td>$5\frac{1}{3}$ (not a whole number)</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$2\frac{2}{3}$ (not a whole number)</td>
</tr>
<tr>
<td>5</td>
<td>$1\frac{1}{3}$ (not a whole number)</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

The ordered pairs from the table that we want to use are (0, 8), (3, 4), and (6, 0). The graph would look like the one shown below:
Based on the number of points in the graph, there are three possible ways that Silas can buy pounds of fudge: 8 pounds of peanut butter only, 3 pounds of chocolate and 4 pounds of peanut butter, or 6 pounds of chocolate only. Notice that if the points on the graph were joined, they would form a line. If Silas allowed himself to buy partial pounds of fudge, then there would be many more possible combinations. Each combination would total $24 and be represented by a point on the line that contains (0, 8), (3, 4), and (6, 0).
Example:
2. Graph the inequality $x + 2y < 4$.

Solution:
The graph looks like a half-plane with a dashed boundary line. The shading is below the line because the points that satisfy the inequality fall below the line. First, graph the line using $x$- and $y$-intercepts. For the $x$-intercept, solve for $y = 0$. For the $y$-intercept, solve for $x = 0$.

\[
\begin{align*}
  x + 2(0) &< 4 \\
  x &< 4 \\
  (0) + 2y &< 4 \\
  2y &< 4 \\
  y &< 2
\end{align*}
\]

This gives the points $(4, 0)$ and $(0, 2)$. Since the inequality used the $<$ symbol, use a dotted line through the two points.

Next, decide which side of the boundary line to shade. Use $(0, 0)$ as a test point. Is $0 + 2(0) < 4$? Yes, so $(0, 0)$ is a solution of the inequality. Shade the region below the line. The graph for $x + 2y < 4$ is represented below.
Build a Function That Models a Relationship between Two Quantities

**MGSE9-12.F.BF.1** Write a function that describes a relationship between two quantities.

**MGSE9-12.F.BF.1a** Determine an explicit expression and the recursive process (steps for calculation) from context. For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression “2x + 15” can be described recursively (either in writing or verbally) as “to find out how much money Jimmy will have tomorrow, you add $2 to his total today.” \( J_n = J_{n-1} + 2, \ J_0 = 15. \)

**MGSE9-12.F.BF.2** Write arithmetic and geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect arithmetic sequences to linear functions and geometric sequences to exponential functions.

**KEY IDEAS**

1. Modeling a quantitative relationship can be a challenge. But there are some techniques we can use to make modeling easier. Functions can be written to represent the relationship between two variables.

   **Example:**

   Joe started with $13. He has been saving $2 each week to purchase a baseball glove. The amount of money Joe has depends on how many weeks he has been saving. This means the money in his bank account is the dependent variable and the number of weeks is the independent variable. So, the number of weeks and the amount Joe has saved are related. We can begin with the function \( S(x) \), where \( S \) is the amount he has saved and \( x \) is the number of weeks. Since we know that he started with \$13\ and that he saves \$2\ each week we can use a linear model, one where the change is constant.

   A linear model for a function is \( f(x) = mx + b \), where \( m \) and \( b \) are any real numbers and \( x \) is the independent variable.

   So, the model is \( S(x) = 2x + 13 \), which will generate the amount Joe has saved after \( x \) weeks.
2. Sometimes the data for a function is presented as a sequence.

Example:
Suppose we know the total number of cookies eaten by Rachel on a day-to-day basis over the course of a week. We might get a sequence like this: 3, 5, 7, 9, 11, 13, 15. There are two ways we could model this sequence. The first would be the explicit way. We would arrange the sequence in a table. Note that \( d \) in the third row means change, or difference.

\[
\begin{array}{cccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  a_n & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\
  d & - & 5 - 3 = 2 & 7 - 5 = 2 & 9 - 7 = 2 & 11 - 9 = 2 & 13 - 11 = 2 & 15 - 13 = 2 \\
\end{array}
\]

Since the difference between successive terms of the sequence is constant, namely 2, we can again use a linear model. But this time we do not know the \( y \)-intercept because there is no zero term \((n = 0)\). However, if we work backward, \( a_0 \)—the term before the first—would be 1, so the starting number would be 1. That leaves us with an explicit formula: \( f(n) = 2n + 1 \), for \( n > 0 \) \((n \text{ is an integer})\). The explicit formula \( a_n = a_1 + d(n - 1) \), where \( a_1 \) is the first term and \( d \) is the common difference, can be used to find the explicit function. A sequence that can be modeled with a linear function is called an arithmetic sequence.

Another way to look at the sequence is recursively. We need to express term \( n \) \((a_n)\) in terms of a previous term \((a_{n-1})\). Since \( n \) is the term, then \( n - 1 \) is used to represent the previous term. For example, \( a_3 \) is the third term, so \( a_{3-1} = a_2 \) is the second term. Since the constant difference is 2, we know \( a_n = a_{n-1} + 2 \) for \( n > 1 \), with \( a_1 = 3 \).
Unit 2: Reasoning with Linear Equations and Inequalities

SAMPLE ITEMS

1. Which function represents the sequence?

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_n</td>
<td>3</td>
<td>10</td>
<td>17</td>
<td>24</td>
<td>31</td>
<td>...</td>
</tr>
</tbody>
</table>

A. \( f(n) = n + 3 \)
B. \( f(n) = 7n - 4 \)
C. \( f(n) = 3n + 7 \)
D. \( f(n) = n + 7 \)

Correct Answer: B

2. Each week, Tim wants to increase the number of sit-ups he does daily by 2 sit-ups. The first week, he does 15 sit-ups each day.

Write an explicit function in the form \( f(n) = mn + b \) to represent the number of sit-ups, \( f(n) \), Tim does daily in week \( n \).

Solution:
The difference between the number of daily sit-ups each week is always 2, so this is a linear model with \( m = 2 \). Since there is no zero term, we take the first term, \( (n = 1) \), and work backwards by subtracting 2 from 15. This gives us \( b = 13 \). Therefore, the explicit function is \( f(n) = 2n + 13 \).

A recursive function in the form \( f(n + 1) = f(n) + d \), where \( f(1) = a \), can be written for the sit-up problem. What recursive function represents the number of sit-ups, \( f(n) \), Tim does daily in week \( n \)?

Solution:
Since Tim starts out doing 15 sit-ups each day, \( f(1) = 15 \). The variable \( d \) stands for the difference between the number of daily sit-ups Tim does each week, which is 2. The recursive function will be \( f(n + 1) = f(n) + 2 \), where \( f(1) = 15 \).
Understand the Concept of a Function and Use Function Notation

**MGSE9-12.F.IF.1** Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e., each input value maps to exactly one output value. If \( f \) is a function, \( x \) is the input (an element of the domain), and \( f(x) \) is the output (an element of the range). Graphically, the graph is \( y = f(x) \).

**MGSE9-12.F.IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

**MGSE9-12.F.IF.3** Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1, 2, 3, 4 . . .) By graphing or calculating terms, students should be able to show how the recursive sequence \( a_1 = 7, a_n = a_{n-1} + 2; \) the sequence \( s_n = 2(n - 1) + 7; \) and the function \( f(x) = 2x + 5 \) (when \( x \) is a natural number) all define the same sequence.

**KEY IDEAS**

1. There are many ways to show how pairs of quantities are related. Some of them are shown below.
   - **Mapping Diagrams**
     
     ![Mapping Diagrams](image)

   - **Sets of Ordered Pairs**
     
     Set I: \{\( (1, 1), (1, 2), (2, 4), (3, 3) \)\}
     
     Set II: \{\( (1, 1), (1, 5), (2, 3), (3, 3) \)\}
     
     Set III: \{\( (1, 1), (2, 3), (3, 5) \)\}

   - **Tables of Values**
     
     | I    | II   | III  |
     |------|------|------|
     | \( x \) | \( y \) | \( x \) | \( y \) | \( x \) | \( y \) |
     | 1    | 1    | 1    | 1    | 1    | 1    |
     | 1    | 2    | 1    | 5    | 1    | 1    |
     | 2    | 4    | 2    | 3    | 2    | 3    |
     | 3    | 3    | 3    | 3    | 3    | 5    |
The relationship shown in Mapping Diagram I, Set I, and Table I all represent the same paired numbers. Likewise, Mapping II, Set II, and Table II all represent the same quantities. The same goes for the third group of displays.

Notice the arrows in the mapping diagrams are all arranged from left to right. The numbers on the left side of the mapping diagrams are the same as the x-coordinates in the ordered pairs as well as the values in the first column of the tables. Those numbers are called the input values of a quantitative relationship and are known as the domain. The numbers on the right of the mapping diagrams, the y-coordinates in the ordered pairs, and the values in the second column of the table are the output, or range. Every number in the domain is assigned to at least one number of the range.

Mapping diagrams, ordered pairs, and tables of values are good to use when there are a limited number of input and output values. There are some instances when the domain has an infinite number of elements to be assigned. In those cases, it is better to use either an algebraic rule or a graph to show how pairs of values are related. Often we use equations as the algebraic rules for the relationships. The domain can be represented by the independent variable and the range can be represented by the dependent variable.

2. A function is a quantitative relationship where each member of the domain is assigned to exactly one member of the range. Of the relationships on the previous page, only III is a function. In I and II, there were members of the domain that were assigned to two elements of the range. In particular, in I, the value 1 of the domain was paired with 1 and 2 of the range. The relationship is a function if two values in the domain are related to the same value in the range.

Consider the vertical line $x = 2$. Every point on the line has the same x-value and a different y-value. So the value of the domain is paired with infinitely many values of the range. This line is not a function. In fact, all vertical lines are not functions.

3. A function can be described using a function rule that represents an output value, or element of the range, in terms of an input value, or element of the domain.
A function rule can be written in **function notation**. Here is an example of a function rule and its notation.

\[ y = 3x + 5 \]
\[ f(x) = 3x + 5 \]
\[ f(2) = 3(2) + 5 \]

Here, \( y \) is the output and \( x \) is the input. Read as “\( f \) of \( x \).” “\( f \) of \( 2 \),” the value of the function at \( x = 2 \), is the output when \( 2 \) is the input.

Be careful—do not confuse the parentheses used in notation with multiplication.

Functions can also represent real-life situations where \( f(15) = 45 \) can represent 15 books that cost $45. Functions can have restrictions or constraints that only include whole numbers, such as the situation of the number of people in a class and the number of books in the class. There cannot be half a person or half a book.

Note that all functions have a corresponding graph. The points that lie on the graph of a function are formed using input values, or elements of the domain as the \( x \)-coordinates, and output values, or elements of the range as the \( y \)-coordinates.

**Example:**
Given \( f(x) = 2x - 1 \), find \( f(7) \).

**Solution:**
\[ f(7) = 2(7) - 1 = 14 - 1 = 13 \]

**Example:**
If \( g(6) = 3 - 5(6) \), what is \( g(x) \)?

**Solution:**
\[ g(x) = 3 - 5x \]

**Example:**
If \( f(-2) = -4(-2) \), what is \( f(b) \)?

**Solution:**
\[ f(b) = -4b \]
Example:
Graph \( f(x) = 2x - 1 \).

Solution:
In the function rule \( f(x) = 2x - 1 \), \( f(x) \) is the same as \( y \).

Then we can make a table of \( x \) (input) and \( y \) (output) values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

The values in the rows of the table form ordered pairs. We plot those ordered pairs. If the domain is not specified, we connect the points. If the numbers in the domain are not specified, we assume that they are all real numbers. If the domain is specified such as whole numbers only, then connecting the points is not needed.
4. A **sequence** is an ordered list of numbers. Each number in the sequence is called a **term**. The terms are consecutive or identified as the first term, second term, third term, and so on. The pattern in the sequence is revealed in the relationship between each term and its term number, or in a term’s relationship to the previous term in the sequence.

**Example:**
Consider the sequence 3, 6, 9, 12, 15, . . . The first term is 3, the second term is 6, the third term is 9, and so on. The “. . .” at the end of the sequence indicates the pattern continues without end. Can this pattern be considered a function?

**Solution:**
There are different ways of seeing a pattern in the sequence above. The initial term (y-intercept) and the slope can be used to create a table to derive the function. One way is to say each number in the sequence is 3 times the number of its term. For example, the fourth term would be 3 times 4, or 12. Looking at the pattern in this way, all you would need to know is the number of the term, and you could predict the value of the term. The value of each term would be a function of its term number. We could use this relationship to write an algebraic rule for the sequence, $y = 3x$, where $x$ is the number of the term and $y$ is the value of the term. This algebraic rule would only assign one number to each input value from the numbers 1, 2, 3, etc. So, we could write a function for the sequence. We can call the function $T$ and write its rule as $T(n) = 3n$, where $n$ is the term number and 3 is the difference between each term in the sequence, called the common difference. The domain for the function $T$ would be counting numbers. The range would be the value of the terms in the sequence. When an equation with the term number as a variable is used to describe a sequence, we refer to it as the **explicit formula** for the sequence, or the **closed form**. We could also use the common difference and the initial term to find the explicit formula by using $a_n = a_1 + d(n - 1)$, where $a_1$ is the first term and $d$ is the common difference. We can create the explicit function $T(n) = 3(n - 1) + 3$ for all $n > 0$. The domain for this function would be natural numbers.

Another way to describe the sequence in the example above is to say each term is three more than the term before it. Instead of using the number of the term, you would need to know a previous term to find a subsequent term’s value. We refer to a sequence represented in this form as a **recursive formula**.

**Important Tips**
- Use language carefully when talking about functions. For example, use $f$ to refer to the function as a whole and use $f(x)$ to refer to the output when the input is $x$.
- Be sure to check all the terms you are provided with before reaching the conclusion that there is a pattern.
REVIEW EXAMPLES

1. A manufacturer keeps track of her monthly costs by using a “cost function” that assigns a total cost for a given number of manufactured items, \( x \). The function is \( C(x) = 5,000 + 1.3x \).

   a. What is the reasonable domain of the function?
   b. What is the cost of 2,000 items?
   c. If costs must be kept below $10,000 this month, what is the greatest number of items she can manufacture?

Solution:
   a. Since \( x \) represents a number of manufactured items, it cannot be negative, nor can a fraction of an item be manufactured. Therefore, the domain can only include values that are whole numbers.
   b. Substitute 2,000 for \( x \): \( C(2,000) = 5,000 + 1.3(2,000) = 7,600 \)
   c. Form an inequality:

\[
C(x) < 10,000 \\
5,000 + 1.3x < 10,000 \\
1.3x < 5,000 \\
x < 3,846.2, \text{ or } 3,846 \text{ items}
\]

2. Consider the first six terms of this sequence: 1, 3, 9, 27, 81, 243, . . .

   a. What is \( a_1 \)? What is \( a_3 \)?
   b. What is the reasonable domain of the function?
   c. If the sequence defines a function, what is the range?
   d. What is the common ratio of the function?

Solution:
   a. \( a_1 \) is 1 and \( a_3 \) is 9.
   b. The domain is the set of counting numbers: \( \{1, 2, 3, 4, 5, \ldots \} \).
   c. The range is \( \{1, 3, 9, 27, 81, 243, \ldots \} \).
   d. The common ratio is 3.
3. The function \( f(n) = -(1 - 4n) \) represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

Solution:

\[
\begin{array}{c|c|c|c|c|c}
  n & 1 & 2 & 3 & 4 & 5 \\
  \hline
  f(n) & 3 & 7 & 11 & 15 & 19 \\
\end{array}
\]

Since the function is a sequence, the domain would be \( n \), the number of each term in the sequence. The set of numbers in the domain can be written as \( \{1, 2, 3, 4, 5, \ldots \} \). Notice that the domain is an infinite set of numbers, even though the table only displays the first five elements.

The range is \( f(n) \) or \( (a_n) \), the output numbers that result from applying the rule \(-1(1 - 4n)\). The set of numbers in the range, which is the sequence itself, can be written as \( \{3, 7, 11, 15, 19, \ldots \} \). This is also an infinite set of numbers, even though the table only displays the first five elements.

**SAMPLE ITEMS**

1. Look at the sequence in this table.

\[
\begin{array}{c|c|c|c|c|c}
  n & 1 & 2 & 3 & 4 & 5 \\
  \hline
  a_n & -1 & 1 & 3 & 5 & 7 \\
\end{array}
\]

Which function represents the sequence?

A. \( a_n = a_{n-1} + 1 \)
B. \( a_n = a_{n-1} + 2 \)
C. \( a_n = 2a_{n-1} - 1 \)
D. \( a_n = 2a_{n-1} - 3 \)

Correct Answer: B
2. Consider this pattern.

Which function represents the sequence that represents the pattern?

A. \( a_n = a_{n-1} - 3 \)
B. \( a_n = a_{n-1} + 3 \)
C. \( a_n = 3a_{n-1} - 3 \)
D. \( a_n = 3a_{n-1} + 3 \)

Correct Answer: B

3. Which function is modeled in this table?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
</tr>
</tbody>
</table>

A. \( f(x) = x + 7 \)
B. \( f(x) = x + 9 \)
C. \( f(x) = 2x + 5 \)
D. \( f(x) = 3x + 5 \)

Correct Answer: D
4. Which explicit formula describes the pattern in this table?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>6.28</td>
</tr>
<tr>
<td>3</td>
<td>9.42</td>
</tr>
<tr>
<td>5</td>
<td>15.70</td>
</tr>
<tr>
<td>10</td>
<td>31.40</td>
</tr>
</tbody>
</table>

A. \( d = 3.14 \times C \)
B. \( 3.14 \times C = d \)
C. \( 31.4 \times 10 = C \)
D. \( C = 3.14 \times d \)

Correct Answer: D

5. If \( f(12) = 4(12) - 20 \), which function gives \( f(x) \)?

A. \( f(x) = 4x \)
B. \( f(x) = 12x \)
C. \( f(x) = 4x - 20 \)
D. \( f(x) = 12x - 20 \)

Correct Answer: C
Interpret Functions That Arise in Applications in Terms of the Context

**MGSE9-12.F.IF.4** Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior.

**MGSE9-12.F.IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

**MGSE9-12.F.IF.6** Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

**KEY IDEAS**

1. By examining the graph of a function, many of its features are discovered. Features include domain and range; x- and y-intercepts; intervals where the function values are increasing, decreasing, positive, or negative; and rates of change.

**Example:**
Consider the graph of \( f(x) = x \). It appears to be an unbroken line and slanted upward.

Some of its key features are
- Domain: All real numbers
- Range: All real numbers
- \( x \)-intercept: The line appears to intersect the \( x \)-axis at 0.
- \( y \)-intercept: The line appears to intersect the \( y \)-axis at 0.
Unit 2: Reasoning with Linear Equations and Inequalities

- Increasing: for $x (-\infty, \infty)$: as $x$ increases, $f(x)$ increases
- Decreasing: Never
- Positive: $f(x) > 0$ when $x > 0$
  Negative: $f(x) < 0$ when $x < 0$
- Rate of change: 1
- End behavior: decreases as $x$ goes to $-\infty$ and increases as $x$ goes to $\infty$

Example:
Consider the graph of $f(x) = -x$. It appears to be an unbroken line and slanted downward.

Some of its key features are
- Domain: All real numbers because there is a point on the graph for every possible $x$-value
- Range: All real numbers because there is a point on the graph that corresponds to every possible $y$-value
- $x$-intercept: It appears to intersect the $x$-axis at 0.
- $y$-intercept: It appears to intersect the $y$-axis at 0.
- Increasing: The function does not increase.
- Decreasing: for $x (-\infty, \infty)$
- Positive: $f(x)$ is positive for $x < 0$
  Negative: $f(x)$ is negative for $x > 0$
- Rate of change: $-1$
- End behavior: increases as $x$ goes to $-\infty$ and decreases as $x$ goes to $\infty$
2. Other features of functions can be discovered through examining their tables of values. The intercepts may appear in a table of values. From the differences of \( f(x) \)-values over various intervals, we can tell if a function grows at a constant rate of change.

**Example:**

Let \( h(x) \) be the number of hours it takes a new factory to produce \( x \) engines. The company’s accountant determines that the number of hours it takes depends on the time it takes to set up the machinery and the number of engines to be completed. It takes 6.5 hours to set up the machinery to make the engines and about 5.25 hours to completely manufacture one engine. The relationship is modeled with the function \( h(x) = 6.5 + 5.25x \). Next, the accountant makes a table of values to check his function against his production records. The accountant starts with 0 engines because of the time it takes to set up the machinery.

The realistic domain for the accountant’s function would be whole numbers because you cannot manufacture a negative number of engines.

<table>
<thead>
<tr>
<th>( x ) (number of engines)</th>
<th>( h(x) ) (hours to produce)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.5</td>
</tr>
<tr>
<td>1</td>
<td>11.75</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>22.25</td>
</tr>
<tr>
<td>4</td>
<td>27.5</td>
</tr>
<tr>
<td>5</td>
<td>32.75</td>
</tr>
<tr>
<td>10</td>
<td>59</td>
</tr>
<tr>
<td>100</td>
<td>531.5</td>
</tr>
</tbody>
</table>

From the table we can see the \( y \)-intercept. The \( y \)-intercept is the \( y \)-value when \( x = 0 \). The very first row of the table indicates the \( y \)-intercept is 6.5. Since we do not see the number 0 in the \( h(x) \) column, we cannot tell from the table whether there is an \( x \)-intercept. The \( x \)-intercept is the value when \( h(x) = 0 \).

\[
\begin{align*}
h(x) &= 6.5 + 5.25x \\
0 &= 6.5 + 5.25x \\
-6.5 &= 5.25x \\
-1.24 &= x
\end{align*}
\]

The \( x \)-value when \( y = 0 \) is negative, which is not possible in the context of this example.
The accountant’s table also gives us an idea of the rate of change of the function. We should notice that as $x$-values are increasing by 1, the $h(x)$-values are growing by increments of 5.25. There appears to be a constant rate of change when the input values increase by the same amount. The increase from both 3 engines to 4 engines and 4 engines to 5 engines is 5.25 hours. The average rate of change can be calculated by comparing the values in the first or last rows of the table. The increase in number of engines manufactured is 100 – 0, or 100. The increase in hours to produce the engines is 531.5 – 6.5, or 525. The average rate of change is $\frac{525}{100} = 5.25$. The units for this average rate of change would be hours/engine, which happens to be the exact amount of time it takes to manufacture 1 engine.

**Important Tips**

- One method for exploration of a new function could begin by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.

- You cannot always find exact values from a graph. Always check your answers using the equation.
REVIEW EXAMPLE

1. A company uses the function \( V(x) = 28,000 - 1,750x \) to represent the amount left to pay on a truck, where \( V(x) \) is the amount left to pay on the truck, in dollars, and \( x \) is the number of months after its purchase. Use the table of values shown below.

<table>
<thead>
<tr>
<th>( x ) (months)</th>
<th>( V(x) ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>28,000</td>
</tr>
<tr>
<td>1</td>
<td>26,250</td>
</tr>
<tr>
<td>2</td>
<td>24,500</td>
</tr>
<tr>
<td>3</td>
<td>22,750</td>
</tr>
<tr>
<td>4</td>
<td>21,000</td>
</tr>
<tr>
<td>5</td>
<td>19,250</td>
</tr>
</tbody>
</table>

a. What is the \( y \)-intercept of the graph of the function in terms of the amount left to pay on the truck?
b. Does the graph of the function have an \( x \)-intercept, and if so what does that represent?
c. Does the function increase or decrease?

Solution:

a. From the table, when \( x = 0 \), \( V(x) = 28,000 \). So, the \( y \)-intercept is 28,000, which means at zero months, the amount left to pay on the truck had not yet decreased.

b. Yes, it does have an \( x \)-intercept, although it is not shown in the table. The \( x \)-intercept is the value of \( x \) when \( V(x) = 0 \).

\[
0 = 28,000 - 1,750x \\
-28,000 = -1,750x \\
16 = x
\]

The \( x \)-intercept is 16. This means that the truck is fully paid off after 16 months of payments.

c. For \( x > 0 \), as \( x \) increases, \( V(x) \) decreases. Therefore, the function decreases.
SAMPLE ITEM

1. A wild horse runs at a rate of 8 miles an hour for 6 hours. Let $y$ be the distance, in miles, the horse travels for a given amount of time, $x$, in hours. This situation can be modeled by a function.

Which of these describes the domain of the function?

A. $0 \leq x \leq 6$
B. $0 \leq y \leq 6$
C. $0 \leq x \leq 48$
D. $0 \leq y \leq 48$

Correct Answer: A
Analyze Functions Using Different Representations

**MGSE9-12.F.IF.7** Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

**MGSE9-12.F.IF.7a** Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

**MGSE9-12.F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.

**KEY IDEA**

1. When working with functions, it is essential to be able to interpret the specific quantitative relationship regardless of the manner of its presentation. Understanding different representations of functions, such as tables, graphs, equations, and verbal descriptions, makes interpreting relationships between quantities easier. Beginning with lines, we will learn how each representation aids our understanding of a function. Almost all lines are functions, except vertical lines, because they assign multiple elements of their range to just one element in their domain. All linear functions can be written in the form $y = mx + b$, where $m$ and $b$ are real numbers and $x$ is a variable to which the function $f$ assigns a corresponding value, $f(x)$.

**Example:**

Consider the linear functions $f(x) = x + 5$, $g(x) = 2x - 5$, and $h(x) = -2x$.

First, we will make a table of values for each equation. To begin, we need to decide on the domains. In theory, $f(x)$, $g(x)$, and $h(x)$ can accept any number as input. So, the three of them have all real numbers as their domains. But for a table, we can only include a few elements of their domains. We should choose a sample that includes negative numbers, 0, and positive numbers. Place the elements of the domain in the left column, usually in ascending order. Then apply the function rule to determine the corresponding elements in the range. Place them in the right column.
### Unit 2: Reasoning with Linear Equations and Inequalities

<table>
<thead>
<tr>
<th>x</th>
<th>( f(x) = x + 5 )</th>
<th>x</th>
<th>( g(x) = 2x - 5 )</th>
<th>x</th>
<th>( h(x) = -2x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>2</td>
<td>-3</td>
<td>-11</td>
<td>-3</td>
<td>6</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
<td>-2</td>
<td>-9</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>-1</td>
<td>-7</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>0</td>
<td>-5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>1</td>
<td>-3</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>-6</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>-8</td>
</tr>
</tbody>
</table>

We can note several features about the functions just from their tables of values.

- \( f(x) \) has a \( y \)-intercept of 5. When \( x \) is 0, \( f(x) = 5 \). It is represented by \((0, 5)\) on its graph.
- \( g(x) \) has a \( y \)-intercept of \(-5\). When \( x \) is 0, \( g(x) = -5 \). It is represented by \((0, -5)\) on its graph.
- \( h(x) \) has a \( y \)-intercept of 0. When \( x \) is 0, \( h(x) = 0 \). It is represented by \((0, 0)\) on its graph.
- \( h(x) \) has an \( x \)-intercept of 0. When \( h(x) = 0 \), \( x = 0 \). It is represented by \((0, 0)\) on its graph.
- \( f(x) \) has an average rate of change of 1. \( \frac{9 - 2}{4 - (-3)} = 1 \)
- \( g(x) \) has an average rate of change of 2. \( \frac{3 - (-11)}{4 - (-3)} = 2 \)
- \( h(x) \) has an average rate of change of \(-2\). \( \frac{-8 - 6}{4 - (-3)} = -2 \)

Now we will take a look at the graphs of \( f(x) \), \( g(x) \), and \( h(x) \).
Their graphs confirm what we already knew about their intercepts and their constant rates of change. To confirm, we can see that \( f(x) \) increases by 2.5, as \( x \) increases by 2.5, which is a \( 1 \) to \( 1 \) rate of change. So the slope of \( f(x) \) is 1. \( g(x) \) increases by 10 as \( x \) increases by 5, which is a \( 2 \) to \( 1 \) rate of change. So the slope is 2. \( h(x) \) decreases by 10 as \( x \) increases by 2, which is a \( -2 \) to \( 1 \) rate of change. So the slope is \( -2 \). The graphs suggest other information:

- \( f(x) \) appears to have positive values for \( x > -5 \) and negative values for \( x < -5 \).
- \( f(x) \) appears to be always increasing with no maximum or minimum values.
- \( g(x) \) appears to have positive values for \( x > 2.5 \) and negative values for \( x < 2.5 \).
- \( g(x) \) appears to be always increasing with no maximum or minimum values.
- \( h(x) \) appears to have positive values for \( x < 0 \) and negative values for \( x > 0 \).
- \( h(x) \) appears to be always decreasing with no maximum or minimum values.

To confirm these observations, we can work with the equations for the functions. We suspect \( f(x) \) is positive for \( x > -5 \). Since \( f(x) \) is positive whenever \( f(x) > 0 \), write and solve the inequality \( x + 5 > 0 \) and solve for \( x \). We get \( f(x) > 0 \) when \( x > -5 \). We can confirm all our observations about \( f(x) \) from working with the equation. Likewise, the observations about \( g(x) \) and \( h(x) \) can be confirmed using their equations.

**Important Tips**

- Remember that the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function. The domain and range can also be determined by examining the graph of a function by looking for asymptotes on the graph of an exponential function or looking for endpoints or continuity for linear and exponential functions.
- Be familiar with important features of a function such as intercepts, domain, range, minimums and maximums, end behavior, asymptotes, and periods of increasing and decreasing values.
REVIEW EXAMPLES

1. What are the key features of the function \( p(x) = \frac{1}{2}x - 3 \)?

   **Solution:**
   
   First, notice that the function is linear. The domain for the function is the possible numbers we can substitute for \( x \). Since the function is linear and is not related to a real-life situation where certain values are not applicable, the domain is all real numbers. The graphic representation will give us a better idea of its range.

   We can determine the \( y \)-intercept by finding \( p(0) \):

   \[
   p(0) = \frac{1}{2}(0) - 3 = -3
   \]

   So, the graph of \( p(x) \) will intersect the \( y \)-axis at \((0, -3)\). To find the \( x \)-intercept, we have to solve the equation \( p(x) = 0 \).

   \[
   \frac{1}{2}x - 3 = 0
   \]

   \[
   \frac{1}{2}x = 3
   \]

   \[
   x = 6
   \]

   So, the \( x \)-intercept is 6. The line intersects the \( x \)-axis at \((6, 0)\).

   Now we will make a table of values to investigate the rate of change of \( p(x) \).
Notice the row that contains the values 0 and –3. These numbers correspond to the point where the line intersects the y-axis, confirming that the y-intercept is –3. Since 0 does not appear in the right column, the coordinates of the x-intercept are not in the table of values. We notice that the values in the right column keep increasing by \( \frac{1}{2} \). We can calculate the average rate of change.

Average rate of change: \( \frac{-1 - (-9)}{4 - (-3)} = \frac{1}{2} \)

It turns out that the average rate of change is the same as the incremental differences in the outputs. This confirms that the function \( p(x) \) has a constant rate of change. Notice that \( \frac{1}{2} \) is the coefficient of \( x \) in the function rule.

Now we will examine the graph. The graph shows a line that appears to be always increasing. Since the line has no minimum or maximum value, its range would be all real numbers. The function appears to have positive values for \( x > 6 \) and negative values for \( x < 6 \).
2. Compare \( p(x) = \frac{1}{2}x - 3 \) from the previous example with the function \( m(x) \) in the graph below.

The graph of \( m(x) \) intersects both the \( x \)- and \( y \)-axes at 0. It appears to have a domain of all real numbers and a range of all real numbers. So, \( m(x) \) and \( p(x) \) have the same domain and range. The graph appears to have a constant rate of change and is decreasing. It has positive values when \( x < 0 \) and negative values when \( x > 0 \).
SAMPLE ITEMS

1. To rent a canoe, the cost is $3 for the oars and life preserver, plus $5 an hour for the canoe. Which graph models the cost of renting a canoe?

Correct Answer: C
2. Juan and Patti decided to see who could read more books in a month. They began to keep track after Patti had already read 5 books that month. This graph shows the number of books Patti read for the next 10 days and the rate at which she will read for the rest of the month.

If Juan does not read any books before day 4 and he starts reading at the same rate as Patti for the rest of the month, how many books will he have read by day 12?

A. 5  
B. 10  
C. 15  
D. 20

Correct Answer: B