Lesson 7: Calculating Probabilities of Compound Events

Classwork

A previous lesson introduced tree diagrams as an effective method of displaying the possible outcomes of certain multistage chance experiments. Additionally, in such situations, tree diagrams were shown to be helpful for computing probabilities.

In those previous examples, diagrams primarily focused on cases with two stages. However, the basic principles of tree diagrams can apply to situations with more than two stages.

Example 1: Three Nights of Games

Recall a previous example where a family decides to play a game each night, and they all agree to use a tetrahedral die (a four-sided die in the shape of a pyramid where each of four possible outcomes is equally likely) each night to randomly determine if the game will be a board (B) or a card (C) game. The tree diagram mapping the possible overall outcomes over two consecutive nights was as follows:

- **Monday**: B, C
- **Tuesday**: B, C
- **Outcome**: BB, BC, CB, CC
But how would the diagram change if you were interested in mapping the possible overall outcomes over three consecutive nights? To accommodate this additional third stage, you would take steps similar to what you did before. You would attach all possibilities for the third stage (Wednesday) to each branch of the previous stage (Tuesday).

Exercises 1–3

1. If BBB represents three straight nights of board games, what does CBB represent?

2. List all outcomes where exactly two board games were played over three days. How many outcomes were there?

3. There are eight possible outcomes representing the three nights. Are the eight outcomes representing the three nights equally likely? Why or why not?
Example 2: Three Nights of Games (with Probabilities)

In the example above, each night’s outcome is the result of a chance experiment (rolling the four-sided die). Thus, there is a probability associated with each night's outcome.

By multiplying the probabilities of the outcomes from each stage, you can obtain the probability for each “branch of the tree.” In this case, you can figure out the probability of each of our eight outcomes.

For this family, a card game will be played if the die lands showing a value of 1, and a board game will be played if the die lands showing a value of 2, 3, or 4. This makes the probability of a board game (B) on a given night 0.75.

Let’s use a tree to examine the probabilities of the outcomes for the three days.

Exercises 4–6

4. Probabilities for two of the eight outcomes are shown. Calculate the approximate probabilities for the remaining six outcomes.
5. What is the probability that there will be exactly two nights of board games over the three nights?

6. What is the probability that the family will play at least one night of card games?

**Exercises 7–10: Three Children**

A neighboring family just welcomed their third child. It turns out that all 3 of the children in this family are girls, and they are not twins or triplets. Suppose that for each birth the probability of a boy birth is 0.5 and the probability of a girl birth is also 0.5. What are the chances of having 3 girls in a family’s first 3 births?

7. Draw a tree diagram showing the eight possible birth outcomes for a family with 3 children (no twins or triplets). Use the symbol B for the outcome of *boy* and G for the outcome of *girl*. Consider the first birth to be the first stage. (Refer to Example 1 if you need help getting started.)

8. Write in the probabilities of each stage’s outcomes in the tree diagram you developed above, and determine the probabilities for each of the eight possible birth outcomes for a family with 3 children (no twins).
9. What is the probability of a family having 3 girls in this situation? Is that greater than or less than the probability of having exactly 2 girls in 3 births?

10. What is the probability of a family of 3 children having at least 1 girl?
Lesson Summary

The use of tree diagrams is not limited to cases of just two stages. For more complicated experiments, tree diagrams are used to organize outcomes and to assign probabilities. The tree diagram is a visual representation of outcomes that involve more than one event.

Problem Set

1. According to the Washington, D.C. Lottery’s website for its Cherry Blossom Double” instant scratch game, the chance of winning a prize on a given ticket is about 17%. Imagine that a person stops at a convenience store on the way home from work every Monday, Tuesday, and Wednesday to buy a Scratcher ticket and plays the game. (Source: http://dclottery.com/games/scratchers/1223/cherry-blossom-doubler.aspx accessed May 27, 2013)
   a. Develop a tree diagram showing the eight possible outcomes of playing over these three days. Call stage one “Monday,” and use the symbols W for a winning ticket and L for a non-winning ticket.
   b. What is the probability that the player will not win on Monday but will win on Tuesday and Wednesday?
   c. What is the probability that the player will win at least once during the 3-day period?

2. A survey company is interested in conducting a statewide poll prior to an upcoming election. They are only interested in talking to registered voters.
   Imagine that 55% of the registered voters in the state are male and 45% are female. Also, consider that the distribution of ages may be different for each group. In this state, 30% of male registered voters are age 18–24, 37% are age 25–64, and 33% are 65 or older. 32% of female registered voters are age 18–24, 26% are age 25–64, and 42% are 65 or older.

   The following tree diagram describes the distribution of registered voters. The probability of selecting a male registered voter age 18–24 is 0.165.

   a. What is the chance that the polling company will select a registered female voter age 65 or older?
   b. What is the chance that the polling company will select any registered voter age 18–24?
Lesson 8: The Difference Between Theoretical Probabilities and Estimated Probabilities

Classwork

Did you ever watch the beginning of a Super Bowl game? After the traditional handshakes, a coin is tossed to determine which team gets to kick-off first. Whether or not you are a football fan, the toss of a fair coin is often used to make decisions between two groups.

Examples 1–9: Why a Coin?

Coins were discussed in previous lessons of this module. What is special about a coin? In most cases, a coin has two different sides: a head side (heads) and a tail side (tails). The sample space for tossing a coin is {heads, tails}. If each outcome has an equal chance of occurring when the coin is tossed, then the probability of getting heads is \( \frac{1}{2} \) or 0.5. The probability of getting tails is also 0.5. Note that the sum of these probabilities is 1.

The probabilities formed using the sample space and what we know about coins are called the theoretical probabilities. Using observed relative frequencies is another method to estimate the probabilities of heads or tails. A relative frequency is the proportion derived from the number of the observed outcomes of an event divided by the total number of outcomes. Recall from earlier lessons that a relative frequency can be expressed as a fraction, a decimal, or a percent. Is the estimate of a probability from this method close to the theoretical probability? The following example investigates how relative frequencies can be used to estimate probabilities.

Beth tosses a coin 10 times and records her results. Here are the results from the 10 tosses:

<table>
<thead>
<tr>
<th>Toss</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tr>
<td>Result</td>
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</tbody>
</table>

The total number of heads divided by the total number of tosses is the relative frequency of heads. It is the proportion of the time that heads occurred on these tosses. The total number of tails divided by the total number of tosses is the relative frequency of tails.
1. Beth started to complete the following table as a way to investigate the relative frequencies. For each outcome, the total number of tosses increased. The total number of heads or tails observed so far depends on the outcome of the current toss. Complete this table for the 10 tosses recorded above.

<table>
<thead>
<tr>
<th>Toss</th>
<th>Outcome</th>
<th>Total number of heads so far</th>
<th>Relative frequency of heads so far (to the nearest hundredth)</th>
<th>Total number of tails so far</th>
<th>Relative frequency of tails so far (to the nearest hundredth)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>( \frac{1}{1} = 1 )</td>
<td>0</td>
<td>( \frac{0}{1} = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>H</td>
<td>2</td>
<td>( \frac{2}{2} = 1 )</td>
<td>0</td>
<td>( \frac{0}{2} = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>T</td>
<td>2</td>
<td>( \frac{2}{3} = 0.67 )</td>
<td>1</td>
<td>( \frac{1}{3} = 0.33 )</td>
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</tbody>
</table>

2. What is the sum of the relative frequency of heads and the relative frequency of tails for each row of the table?
3. Beth’s results can also be displayed using a graph. From the table above, complete the graph below using the values of relative frequency of heads so far.

![Graph showing the relative frequency of heads over the number of tosses.]

4. Beth continued tossing the coin and recording results for a total of 40 tosses. Here are the results of the next 30 tosses:

<table>
<thead>
<tr>
<th>Toss</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
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<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result</td>
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<table>
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<tr>
<th>Toss</th>
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<tr>
<td>Result</td>
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</table>

<table>
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<tr>
<th>Toss</th>
<th>31</th>
<th>32</th>
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<th>34</th>
<th>35</th>
<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result</td>
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<td>H</td>
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<td>T</td>
</tr>
</tbody>
</table>

As the number of tosses increases, the relative frequency of heads changes. Complete the following table for the 40 coin tosses:

<table>
<thead>
<tr>
<th>Number of tosses</th>
<th>Total number of heads so far</th>
<th>Relative frequency of heads so far (to the nearest hundredth)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>5</td>
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<td>40</td>
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</tbody>
</table>
5. From the table above, complete the graph below using the relative frequency of heads so far for the total number of tosses of 1, 5, 10, 15, 20, 25, 30, 35, and 40.

6. What do you notice about the changes in the relative frequency of the number of heads so far as the number of tosses increases?

7. If you tossed the coin 100 times, what do you think the relative frequency of heads would be? Explain your answer.

8. Based on the graph and the relative frequencies, what would you estimate the probability of getting heads to be? Explain your answer.
9. How close is your estimate in Exercise 8 to the theoretical probability of 0.5? Would the estimate of this probability have been as good if Beth had only tossed the coin a few times instead of 40?

The value you gave in Exercise 8 is an estimate of the theoretical probability and is called an experimental or estimated probability.

**Exercises 1–8**

Beth received nine more pennies. She securely taped them together to form a small stack. The top penny of her stack showed heads, and the bottom penny showed tails. If Beth tosses the stack, what outcomes could she observe?

1. Beth wanted to determine the probability of getting heads when she tosses the stack. Do you think this probability is the same as the probability of getting heads with just one coin? Explain your answer.

2. Make a sturdy stack of 10 pennies in which one end of the stack has a penny showing heads and the other end tails. Make sure the pennies are taped securely, or you may have a mess when you toss the stack. Toss the stack to observe possible outcomes. What is the sample space for tossing a stack of 10 pennies taped together? Do you think the probability of each outcome of the sample space is equal? Explain your answer.
3. Record the results of 10 tosses. Complete the following table of the relative frequencies of heads for your 10 tosses:

<table>
<thead>
<tr>
<th>Toss</th>
<th>1</th>
<th>2</th>
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4. Based on the value of the relative frequencies of heads so far, what would you estimate the probability of getting heads to be?

5. Toss the stack of 10 pennies another 20 times. Complete the following table:

<table>
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<tr>
<th>Toss</th>
<th>11</th>
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<td>Result</td>
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</table>

6. Summarize the relative frequency of heads so far by completing the following table:

<table>
<thead>
<tr>
<th>Number of tosses</th>
<th>Total number of heads so far</th>
<th>Relative frequency of heads so far (to the nearest hundredth)</th>
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<tbody>
<tr>
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</table>
7. Based on the relative frequencies for the 30 tosses, what is your estimate of the probability of getting heads? Can you compare this estimate to a theoretical probability like you did in the first example? Explain your answer.

8. Create another stack of pennies. Consider creating a stack using 5 pennies, 15 pennies, or 20 pennies taped together in the same way you taped the pennies to form a stack of 10 pennies. Again, make sure the pennies are taped securely, or you might have a mess!

   Toss the stack you made 30 times. Record the outcome for each toss:

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<th>Toss</th>
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Lesson Summary

- Observing the long-run relative frequency of an event from a chance experiment (or the proportion of an event derived from a long sequence of observations) approximates the theoretical probability of the event.
- After a long sequence of observations, the observed relative frequencies get close to the probability of the event occurring.
- When it is not possible to compute the theoretical probabilities of chance experiments, then the long-run relative frequencies (or the proportion of events derived from a long sequence of observations) can be used as estimated probabilities of events.

Problem Set

1. If you created a stack of 15 pennies taped together, do you think the probability of getting a heads on a toss of the stack would be different than for a stack of 10 pennies? Explain your answer.

2. If you created a stack of 20 pennies taped together, what do you think the probability of getting a heads on a toss of the stack would be? Explain your answer.

3. Based on your work in this lesson, complete the following table of the relative frequencies of heads for the stack you created:

<table>
<thead>
<tr>
<th>Number of tosses</th>
<th>Total number of heads so far</th>
<th>Relative frequency of heads so far (to the nearest hundredth)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</table>

4. What is your estimate of the probability that your stack of pennies will land heads up when tossed? Explain your answer.

5. Is there a theoretical probability you could use to compare to the estimated probability? Explain your answer.
Lesson 9: Comparing Estimated Probabilities to Probabilities Predicted by a Model

Classwork

Exploratory Challenge: Game Show—Picking Blue!

Imagine, for a moment, the following situation: You and your classmates are contestants on a quiz show called Picking Blue! There are two bags in front of you, Bag A and Bag B. Each bag contains red and blue chips. You are told that one of the bags has exactly the same number of blue chips as red chips. But you are told nothing about the ratio of blue to red chips in the other bag.

Each student in your class will be asked to select either Bag A or Bag B. Starting with Bag A, a chip is randomly selected from the bag. If a blue chip is drawn, all of the students in your class who selected Bag A win a Blue Token. The chip is put back in the bag. After mixing up the chips in the bag, another chip is randomly selected from the bag. If the chip is blue, the students who picked Bag A win another Blue Token. After the chip is placed back into the bag, the process continues until a red chip is picked. When a red chip is picked, the game moves to Bag B. A chip from the Bag B is then randomly selected. If it is blue, all of the students who selected Bag B win a Blue Token. But if the chip is red, the game is over. Just like for Bag A, if the chip is blue, the process repeats until a red chip is picked from the bag. When the game is over, the students with the greatest number of Blue Tokens are considered the winning team.

Without any information about the bags, you would probably select a bag simply by guessing. But surprisingly, the show’s producers are going to allow you to do some research before you select a bag. For the next 20 minutes, you can pull a chip from either one of the two bags, look at the chip, and then put the chip back in the bag. You can repeat this process as many times as you want within the 20 minutes. At the end of 20 minutes, you must make your final decision and select which of the bags you want to use in the game.

Getting Started

Assume that the producers of the show do not want to give away a lot of their Blue Tokens. As a result, if one bag has the same number of red and blue chips, do you think the other bag would have more, or fewer, blue chips than red chips? Explain your answer.
Planning the Research

Your teacher will provide you with two bags labeled A and B. You have 20 minutes to experiment with pulling chips one at a time from the bags. After you examine a chip, you must put it back in the bag. Remember, no peeking in the bags as that will disqualify you from the game. You can pick chips from just one bag, or you can pick chips from one bag and then the other bag.

Use the results from 20 minutes of research to determine which bag you will choose for the game.

Provide a description outlining how you will carry out your research:

Carrying Out the Research

Share your plan with your teacher. Your teacher will verify whether your plan is within the rules of the quiz show. Approving your plan does not mean, however, that your teacher is indicating that your research method offers the most accurate way to determine which bag to select. If your teacher approves your research, carry out your plan as outlined. Record the results from your research, as directed by your teacher.

Playing the Game

After the research has been conducted, the competition begins. First, your teacher will shake up Bag A. A chip is selected. If the chip is blue, all students who selected Bag A win an imaginary Blue Token. The chip is put back in the bag, and the process continues. When a red chip is picked from Bag A, students selecting Bag A have completed the competition. Your teacher will now shake up Bag B. A chip is selected. If it is blue, all students who selected Bag B win an imaginary Blue Token. The process continues until a red chip is picked. At that point, the game is over.

How many Blue Tokens did you win?
Examining Your Results

At the end of the game, your teacher will open the bags and reveal how many blue and red chips were in each bag. Answer the questions that follow. After you have answered these questions, discuss them with your class.

1. Before you played the game, what were you trying to learn about the bags from your research?

2. What did you expect to happen when you pulled chips from the bag with the same number of blue and red chips? Did the bag that you thought had the same number of blue and red chips yield the results you expected?

3. How confident were you in predicting which bag had the same number of blue and red chips? Explain.

4. What bag did you select to use in the competition and why?

5. If you were the show’s producers, how would you make up the second bag? (Remember, one bag has the same number of red and blue chips.)

6. If you picked a chip from Bag B 100 times and found that you picked each color exactly 50 times, would you know for sure that bag B was the one with equal numbers of each color?
Lesson Summary

- The long-run relative frequencies can be used as estimated probabilities of events.
- Collecting data on a game or chance experiment is one way to estimate the probability of an outcome.
- The more data collected on the outcomes from a game or chance experiment, the closer the estimates of the probabilities are likely to be the actual probabilities.

Problem Set

Jerry and Michael played a game similar to *Picking Blue!*. The following results are from their research using the same two bags:

<table>
<thead>
<tr>
<th></th>
<th>Number of red chips picked</th>
<th>Number of blue chips picked</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jerry’s research:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bag A</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Bag B</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td><strong>Michael’s research:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bag A</td>
<td>28</td>
<td>12</td>
</tr>
<tr>
<td>Bag B</td>
<td>22</td>
<td>18</td>
</tr>
</tbody>
</table>

1. If all you knew about the bags were the results of Jerry’s research, which bag would you select for the game? Explain your answer.
2. If all you knew about the bags were the results of Michael’s research, which bag would you select for the game? Explain your answer.
3. Does Jerry’s research or Michael’s research give you a better indication of the make-up of the blue and red chips in each bag? Explain why you selected this research.
4. Assume there are 12 chips in each bag. Use either Jerry’s or Michael’s research to estimate the number of red and blue chips in each bag. Then, explain how you made your estimates.

   - **Bag A**
     - Number of red chips:
     - Number of blue chips:
   - **Bag B**
     - Number of red chips:
     - Number of blue chips:

5. In a different game of *Picking Blue!*, two bags each contain red, blue, green, and yellow chips. One bag contains the same number of red, blue, green, and yellow chips. In the second bag, half of the chips are blue. Describe a plan for determining which bag has more blue chips than any of the other colors.
Lesson 10: Conducting a Simulation to Estimate the Probability of an Event

Classwork

In previous lessons, you estimated probabilities of events by collecting data empirically or by establishing a theoretical probability model. There are real problems for which those methods may be difficult or not practical to use. Simulation is a procedure that will allow you to answer questions about real problems by running experiments that closely resemble the real situation.

It is often important to know the probabilities of real-life events that may not have known theoretical probabilities. Scientists, engineers, and mathematicians design simulations to answer questions that involve topics such as diseases, water flow, climate changes, or functions of an engine. Results from the simulations are used to estimate probabilities that help researchers understand problems and provide possible solutions to these problems.

Example 1: Families

How likely is it that a family with three children has all boys or all girls?

Let’s assume that a child is equally likely to be a boy or a girl. Instead of observing the result of actual births, a toss of a fair coin could be used to simulate a birth. If the toss results in heads (H), then we could say a boy was born; if the toss results in tails (T), then we could say a girl was born. If the coin is fair (i.e., heads and tails are equally likely), then getting a boy or a girl is equally likely.

Exercises 1–2

Suppose that a family has three children. To simulate the genders of the three children, the coin or number cube or a card would need to be used three times, once for each child. For example, three tosses of the coin resulted in HHT, representing a family with two boys and one girl. Note that HTH and THH also represent two boys and one girl.

1. Suppose that when a prime number (P) is rolled on the number cube, it simulates a boy birth, and a non-prime (N) simulates a girl birth. Using such a number cube, list the outcomes that would simulate a boy birth, and those that simulate a girl birth. Are the boy and girl birth outcomes equally likely?
2. Suppose that one card is drawn from a regular deck of cards. A red card (R) simulates a boy birth, and a black card (B) simulates a girl birth. Describe how a family of three children could be simulated.

Example 2

Simulation provides an estimate for the probability that a family of three children would have three boys or three girls by performing three tosses of a fair coin many times. Each sequence of three tosses is called a trial. If a trial results in either HHH or TTT, then the trial represents all boys or all girls, which is the event that we are interested in. These trials would be called a success. If a trial results in any other order of H’s and T’s, then it is called a failure.

The estimate for the probability that a family has either three boys or three girls based on the simulation is the number of successes divided by the number of trials. Suppose 100 trials are performed, and that in those 100 trials, 28 resulted in either HHH or TTT. Then, the estimated probability that a family of three children has either three boys or three girls would be \( \frac{28}{100} = 0.28 \).

Exercises 3–5

3. Find an estimate of the probability that a family with three children will have exactly one girl using the following outcomes of 50 trials of tossing a fair coin three times per trial. Use H to represent a boy birth and T to represent a girl birth.

HHT HTH HHH TTH THT HTT HHH TTH HHH
HHT TTT HHT TTH HHH HTH THH TTT THT THT
THT HHH THH HTT HTH TTT HHH TTH THT
THH HHT TTT TTH HTT HTH TTT HHH
HTH HTH THT TTH TTT HHT HHT THT TTT HTH
4. Perform a simulation of 50 trials by rolling a fair number cube in order to find an estimate of the probability that a family with three children will have exactly one girl.
   a. Specify what outcomes of one roll of a fair number cube will represent a boy and what outcomes will represent a girl.
   b. Simulate 50 trials, keeping in mind that one trial requires three rolls of the number cube. List the results of your 50 trials.
   c. Calculate the estimated probability.

5. Calculate the theoretical probability that a family with three children will have exactly one girl.
   a. List the possible outcomes for a family with three children. For example, one possible outcome is BBB (all three children are boys).
   b. Assume that having a boy and having a girl are equally likely. Calculate the theoretical probability that a family with three children will have exactly one girl.
Lesson 10: Conducting a Simulation to Estimate the Probability of an Event

Example 3: Basketball Player

Suppose that, on average, a basketball player makes about three out of every four foul shots. In other words, she has a 75% chance of making each foul shot she takes. Since a coin toss produces equally likely outcomes, it could not be used in a simulation for this problem.

Instead, a number cube could be used by specifying that the numbers 1, 2, or 3 represent a hit, the number 4 represents a miss, and the numbers 5 and 6 would be ignored. Based on the following 50 trials of rolling a fair number cube, find an estimate of the probability that she makes five or six of the six foul shots she takes.

| 441323 | 342124 | 442123 | 422313 | 441243 |
| 124144 | 333434 | 243122 | 232323 | 224341 |
| 121411 | 321341 | 111422 | 114232 | 414411 |
| 344221 | 222442 | 343123 | 122111 | 322131 |
| 131224 | 213344 | 321241 | 311214 | 241131 |
| 143143 | 243224 | 323443 | 324243 | 214322 |
| 214411 | 423221 | 311423 | 142141 | 411312 |
| 343214 | 123131 | 242124 | 141132 | 343122 |
| 121142 | 321442 | 121423 | 443431 | 214433 |
| 331113 | 311313 | 211411 | 433434 | 323314 |
Lesson Summary

In previous lessons, you estimated probabilities by collecting data and found theoretical probabilities by creating a model. In this lesson, you used simulation to estimate probabilities in real problems and in situations for which empirical or theoretical procedures are not easily calculated.

Simulation is a method that uses an artificial process (like tossing a coin or rolling a number cube) to represent the outcomes of a real process that provides information about the probability of events. In several cases, simulations are needed to both understand the process as well as provide estimated probabilities.

Problem Set

1. A mouse is placed at the start of the maze shown below. If it reaches station B, it is given a reward. At each point where the mouse has to decide which direction to go, assume that it is equally likely to go in either direction. At each decision point 1, 2, 3, it must decide whether to go left (L) or right (R). It cannot go backwards.

   ![Maze Diagram]

   a. Create a theoretical model of probabilities for the mouse to arrive at terminal points A, B, and C.
      i. List the possible paths of a sample space for the paths the mouse can take. For example, if the mouse goes left at decision point 1, and then right at decision point 2, then the path would be denoted LR.
      ii. Are the paths in your sample space equally likely? Explain.
      iii. What are the theoretical probabilities that a mouse reaches terminal points A, B, and C? Explain.

   b. Based on the following set of simulated paths, estimate the probabilities that the mouse arrives at points A, B, and C.

   RR  RR  RL  LL  LR  RL  LR  LL  LR  RR
   LR  RL  LR  RR  RL  LR  RR  LL  RL  RL
   LL  LR  LL  RR  RR  RL  LL  RR  LR  RL
   RR  LR  RR  LR  LL  LR  RL  RL  LL

   c. How do the simulated probabilities in part (b) compare to the theoretical probabilities of part (a)?
Lesson 10: Conducting a Simulation to Estimate the Probability of an Event

2. Suppose that a dartboard is made up of the $8 \times 8$ grid of squares shown below. Also, suppose that when a dart is thrown, it is equally likely to land on any one of the 64 squares. A point is won if the dart lands on one of the 16 black squares. Zero points are earned if the dart lands in a white square.

![Dartboard Diagram]

- **a.** For one throw of a dart, what is the probability of winning a point? Note that a point is won if the dart lands on a black square.

- **b.** Lin wants to use a number cube to simulate the result of one dart. She suggests that 1 on the number cube could represent a win. Getting 2, 3, or 4 could represent no point scored. She says that she would ignore getting a 5 or 6. Is Lin’s suggestion for a simulation appropriate? Explain why you would use it, or if not, how you would change it.

- **c.** Suppose a game consists of throwing a dart three times. A trial consists of three rolls of the number cube. Based on Lin’s suggestion in part (b) and the following simulated rolls, estimate the probability of scoring two points in three darts.

  | 324 | 332 | 411 | 322 | 124 |
  | 224 | 221 | 241 | 111 | 223 |
  | 321 | 332 | 112 | 433 | 412 |
  | 443 | 322 | 424 | 412 | 433 |
  | 144 | 322 | 421 | 414 | 111 |
  | 242 | 244 | 222 | 331 | 224 |
  | 113 | 223 | 333 | 414 | 212 |
  | 431 | 233 | 314 | 212 | 241 |
  | 421 | 222 | 222 | 112 | 113 |
  | 212 | 413 | 341 | 442 | 324 |

- **d.** The theoretical probability model for winning 0, 1, 2, and 3 points in three throws of the dart as described in this problem is
  
  - i. Winning 0 points has a probability of 0.42;
  - ii. Winning 1 point has a probability of 0.42;
  - iii. Winning 2 points has a probability of 0.14;
  - iv. Winning 3 points has a probability of 0.02.

  Use the simulated rolls in part (c) to build a model of winning 0, 1, 2, and 3 points, and compare it to the theoretical model.
Lesson 11: Conducting a Simulation to Estimate the Probability of an Event

Classwork

Examples 1–2: Simulation

In the last lesson, we used coins, number cubes, and cards to carry out simulations. Another option is putting identical pieces of paper or colored disks into a container, mixing them thoroughly, and then choosing one.

For example, if a basketball player typically makes five out of eight foul shots, then a colored disk could be used to simulate a foul shot. A green disk could represent a made shot, and a red disk could represent a miss. You could put five green and three red disks in a container, mix them, and then choose one to represent a foul shot. If the color of the disk is green, then the shot is made. If the color of the disk is red, then the shot is missed. This procedure simulates one foul shot.

1. Using colored disks, describe how one at-bat could be simulated for a baseball player who has a batting average of 0.300. Note that a batting average of 0.300 means the player gets a hit (on average) three times out of every ten times at bat. Be sure to state clearly what a color represents.

2. Using colored disks, describe how one at-bat could be simulated for a player who has a batting average of 0.273. Note that a batting average of 0.273 means that on average, the player gets 273 hits out of 1,000 at-bats.
Lesson 11: Conducting a Simulation to Estimate the Probability of an Event

Example 3: Using Random Number Tables

Why is using colored disks not practical for the situation described in Example 2? Another way to carry out a simulation is to use a random number table, or a random number generator. In a random number table, the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 occur equally often in the long run. Pages and pages of random numbers can be found online.

For example, here are three lines of random numbers. The space after every five digits is only for ease of reading. Ignore the spaces when using the table.

```
25256 65205 72597 00562 12683 90674 78923 96568 32177 33855
76635 92290 88864 72794 14333 79019 05943 77510 74051 87238
07895 86481 94036 12749 24005 80718 13144 66934 54730 77140
```

To use the random number table to simulate an at-bat for the 0.273 hitter in Exercise 2, you could use a three-digit number to represent one at bat. The three-digit numbers from 000–272 could represent a hit, and the three-digit numbers from 273–999 could represent a non-hit. Using the random numbers above and starting at the beginning of the first line, the first three-digit random number is 252, which is between 000 and 272, so that simulated at-bat is a hit. The next three-digit random number is 566, which is a non-hit.

Continuing on the first line of the random numbers above, what would the hit/non-hit outcomes be for the next six at-bats? Be sure to state the random number and whether it simulates a hit or non-hit.

Example 4: Baseball Player

A batter typically gets to bat four times in a ballgame. Consider the 0.273 hitter from the previous example. Use the following steps (and the random numbers shown above) to estimate that player’s probability of getting at least three hits (three or four) in four times at-bat.

a. Describe what one trial is for this problem.

b. Describe when a trial is called a success and when it is called a failure.

c. Simulate 12 trials. (Continue to work as a class, or let students work with a partner.)
d. Use the results of the simulation to estimate the probability that a 0.273 hitter gets three or four hits in four times at-bat. Compare your estimate with other groups.

Example 5: Birth Month

In a group of more than 12 people, is it likely that at least two people, maybe more, will have the same birth month? Why? Try it in your class.

Now, suppose that the same question is asked for a group of only seven people. Are you likely to find some groups of seven people in which there is a match but other groups in which all seven people have different birth months? In the following exercise, you will estimate the probability that at least two people in a group of seven were born in the same month.

Exercises 1–4

1. What might be a good way to generate outcomes for the birth month problem—using coins, number cubes, cards, spinners, colored disks, or random numbers?

2. How would you simulate one trial of seven birth months?
3. How is a success determined for your simulation?

4. How is the simulated estimate determined for the probability that at least two in a group of seven people were born in the same month?
Lesson Summary

In the previous lesson, you carried out simulations to estimate a probability. In this lesson, you had to provide parts of a simulation design. You also learned how random numbers could be used to carry out a simulation.

To design a simulation:

- Identify the possible outcomes, and decide how to simulate them, using coins, number cubes, cards, spinners, colored disks, or random numbers.
- Specify what a trial for the simulation will look like and what a success and a failure would mean.
- Make sure you carry out enough trials to ensure that the estimated probability gets closer to the actual probability as you do more trials. There is no need for a specific number of trials at this time; however, you want to make sure to carry out enough trials so that the relative frequencies level off.

Problem Set

1. A model airplane has two engines. It can fly if one engine fails but is in serious trouble if both engines fail. The engines function independently of one another. On any given flight, the probability of a failure is 0.10 for each engine. Design a simulation to estimate the probability that the airplane will be in serious trouble the next time it goes up.
   a. How would you simulate the status of an engine?
   b. What constitutes a trial for this simulation?
   c. What constitutes a success for this simulation?
   d. Carry out 50 trials of your simulation, list your results, and calculate an estimate of the probability that the airplane will be in serious trouble the next time it goes up.

2. In an effort to increase sales, a cereal manufacturer created a really neat toy that has six parts to it. One part is put into each box of cereal. Which part is in a box is not known until the box is opened. You can play with the toy without having all six parts, but it is better to have the complete set. If you are really lucky, you might only need to buy six boxes to get a complete set. But if you are very unlucky, you might need to buy many, many boxes before obtaining all six parts.
   a. How would you represent the outcome of purchasing a box of cereal, keeping in mind that there are six different parts? There is one part in each box.
   b. What constitutes a trial in this problem?
   c. What constitutes a success in a trial in this problem?
   d. Carry out 15 trials, list your results, and compute an estimate of the probability that it takes the purchase of 10 or more boxes to get all six parts.
3. Suppose that a type A blood donor is needed for a certain surgery. Carry out a simulation to answer the following question: If 40% of donors have type A blood, what is an estimate of the probability that it will take at least four donors to find one with type A blood?

a. How would you simulate a blood donor having or not having type A?

b. What constitutes a trial for this simulation?

c. What constitutes a success for this simulation?

d. Carry out 15 trials, list your results, and compute an estimate for the probability that it takes at least four donors to find one with type A blood.